

EMBEDDING OF CATEGORY OF TWISTED CHOW-WITT CORRESPONDENCES INTO GEOMETRIC STABLE \mathbb{A}^1 -DERIVED CATEGORY OVER A FIELD

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ABSTRACT. We introduce in this note the notion of the category of twisted Chow-Witt correspondences $CHW(k)$ over a field k of characteristic different from 2. Moreover, we show that over an infinite perfect field this category $CHW(k)_{\mathbb{Q}}$ admits a fully faithful embedding into the geometric stable \mathbb{A}^1 -derived category $D_{\mathbb{A}^1, gm}(k)_{\mathbb{Q}}$, if we assume a conjectural cancellation in the S^1 -stable motivic homotopy category $\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)$ and a conjecture of Morel on the decomposition of stable \mathbb{A}^1 -cohomology. If moreover, -1 is a sum of squares in k , then the result is unconditional.

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1. INTRODUCTION

One of the main motivations for this work is the embedding theorem of Voevodsky [Voe00], which asserts that there is a fully faithful embedding of the category of Grothendieck-Chow pure motives $\underline{Chow}(k)$ into the category of geometric motives $\mathbf{DM}_{gm}(k)$, hence also into the category of motives $\mathbf{DM}_{Nis}^-(k)$

$$\underline{Chow}(k)^{op} \rightarrow \mathbf{DM}_{gm}(k),$$

if k is a perfect field, which admits resolution of singularities (see e.g. [MVW06, Prop. 20.1 and Rem. 20.2]). In this note, we construct a category $CHW(k)$, which we call the category of Chow-Witt correspondences over a field k of characteristic different from 2 and show

Date: 01. 09. 2012.

1991 Mathematics Subject Classification. 14F22, 14F42.

Key words and phrases. stable \mathbb{A}^1 -homotopy, stable \mathbb{A}^1 -cohomology, Chow-Witt correspondences, Milnor-Witt K -theory.

that $CHW(k)_\mathbb{Q}$ admits a fully faithful embedding into the geometric \mathbb{P}^1 -stable \mathbb{A}^1 -derived category $D_{\mathbb{A}^1, gm}(k)_\mathbb{Q}$ under certain conditions. Our work can be viewed as an \mathbb{A}^1 -version for Voevodsky's embedding theorem. The advantage here is that by using duality formalism for \mathbb{P}^1 -stable \mathbb{A}^1 -derived category $D_{\mathbb{A}^1}(k)$ established by A. Asok and C. Haesemeyer in [AH11] (see [Hu05, App. A] for stable \mathbb{A}^1 -homotopy categories), we do not have to assume the resolution of singularities. However, we will systematically avoid to abuse the notion Chow-Witt motives, since we neither consider its idempotent completion nor invert the Lefschetz object. Moreover, unlike in motivic setting, the main problem here is that we don't have cancellation theorem for the effective \mathbb{A}^1 -derived category in general, see [AH11, Rem. 3.2.4], that is the reason why we have to assume the following conjectural cancellation of F. Morel:

Conjecture 1.1. [Mor04, Conj. 6.4.3] *For any S^1 -spectra E and F and any $n \geq 1$, one has*

$$\begin{aligned} \mathrm{Hom}_{\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)}(E \wedge \Sigma_s^\infty \mathbb{G}_m, F \wedge \Sigma_s^\infty \mathbb{G}_m) &\cong \mathrm{Hom}_{\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)}(E \wedge \Sigma_s^\infty \mathbb{G}_m^{\wedge n}, F \wedge \Sigma_s^\infty \mathbb{G}_m^{\wedge n}) \\ &\cong \mathrm{Hom}_{\mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)}(\sigma_{\mathbb{G}_m}(E), \sigma_{\mathbb{G}_m}(F)), \end{aligned}$$

where $\sigma_{\mathbb{G}_m}$ denotes the \mathbb{G}_m -suspension spectrum.

The conjecture is suggested by F. Morel relying on his computation of endomorphism ring of sphere spectrum. Moreover, if we take \mathbb{Q} -coefficient into account, we have equivalences of categories

$$\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)_\mathbb{Q} \simeq D_{\mathbb{A}^1}^{eff}(k)_\mathbb{Q}, \quad \mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)_\mathbb{Q} \simeq D_{\mathbb{A}^1}(k)_\mathbb{Q}.$$

A result of F. Morel (cf. [CD10, Cor. 15.2.14]) said that if -1 is a sum of squares in k , then $D_{\mathbb{A}^1}(k)_\mathbb{Q} \simeq DM_B(k)$, where $DM_B(k)$ denotes the category of Beilinson's motives. Furthermore, F. Morel conjectured in general that:

Conjecture 1.2. [Mor04] *If -1 is not a sum of square in k , then one has a direct decomposition*

$$[S^i, \mathbb{G}_m^{\wedge j}]_{\mathbb{P}^1} \otimes \mathbb{Q} = H_B^{j-i}(k, \mathbb{Q}(j)) \oplus H_{Nis}^{-i}(k, \mathbf{W} \otimes \mathbb{Q}),$$

where $H_B^*(-, \mathbb{Q}(*))$ denotes the Beilinson motivic cohomology and \mathbf{W} is the unramified Witt sheaf.

On the other hand, our interests started originally from the study of the existence of 0-cycles of degree one on algebraic varieties. More precisely, Hélène Esnault asked (cf. [Lev10]): Given a smooth projective variety X over a field k , such that X has a zero cycle of degree one. Are there "motivic" explanations which give the (non)-existence of a k -rational point? In [AH11], A. Asok and C. Haesemeyer show that the existence of zero cycles of degree one over an infinite perfect field of $\mathrm{char}(k) \neq 2$ is equivalent to the assertion that the structure map $\mathbf{H}_0^{st\mathbb{A}^1}(X) \rightarrow \mathbf{H}_0^{st\mathbb{A}^1}(\mathrm{Spec} k)$ is a split epimorphism, where $\mathbf{H}_i^{st\mathbb{A}^1}(X)$ denotes the \mathbb{P}^1 -stable \mathbb{A}^1 -homology sheaves, while in an earlier work [AH11a] they also showed that the existence of a k -rational point over an arbitrary field k is equivalent to the condition that the structure map $\mathbf{H}_0^{\mathbb{A}^1}(X) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(\mathrm{Spec} k)$ is split surjective. So roughly speaking, the obstruction to the lifting of a zero cycle of degree one to a rational point arises by passing from S^1 -spectra to \mathbb{P}^1 -spectra. As remarked by M. Levine, it is not to expect that the category of Chow-Witt correspondences $CHW(k)$ contains any information about the existence of rational points, even assuming the two conjectures above. Now we state our main theorem in this work:

Theorem 1.3. *Let k be a field of $\text{char}(k) \neq 2$. There exists a category $CHW(k)$, whose objects are such a pair $(X, \omega_{X/k})$, where $X \in \text{SmProj}/k$ and $\omega_{X/k}$ denotes its canonical line bundle. The morphisms in $CHW(k)$ are given by*

$$\text{Hom}_{CHW(k)}(X, Y) = \widetilde{\text{CH}}^{\dim(X)}(X \times Y, p_X^{XY*} \omega_{X/k}).$$

If k is an infinite perfect field of $\text{char}(k) \neq 2$, then under the conjectures 1.1 and 1.2 one has a fully faithful embedding

$$CHW(k)_{\mathbb{Q}}^{op} \rightarrow D_{\mathbb{A}^1, gm}(k)_{\mathbb{Q}}.$$

The result on the embedding with \mathbb{Q} -coefficient holds unconditionally, if -1 is a sum of squares in k .

In fact, one of the main steps in the work of [AH11] is to exhibit a natural isomorphism $H_0^{st\mathbb{A}^1}(X)(L) \rightarrow \widetilde{\text{CH}}_0(X_L)$ for any separable, finitely generated field extension L/k . So one may relate this step to our work as evaluating at a generic point. Now our paper is organized as follows: we will review shortly \mathbb{A}^1 -homotopy theory in section §2. Section §3 is devoted for \mathbb{A}^1 -derived categories, in fact we will define the geometric \mathbb{P}^1 -stable \mathbb{A}^1 -derived category $D_{\mathbb{A}^1, gm}(k)$ over a field k in 3.9 at the end of §3. In these §2 and §3 we simply steal everything which is needed from the presentation of [AH11]. For a complete treatment we strongly recommend the reader to [Ay08], [CD10] and [Mor12]. The proof of Theorem 1.3 is divided in two steps. Firstly, we construct $CHW(k)$ in section §4 (see Proposition 4.4 and Def. 4.5). The embedding functor is established then in section §5 (see Proposition 5.5). We fix now some notations throughout this work. For a pair of adjoint functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$, we will adopt the notation in [CD10]

$$F : \mathcal{A} \rightleftarrows \mathcal{B} : G,$$

where F is left adjoint to G and G is right adjoint to F . The unit $1 \rightarrow GF$ will be denoted by $ad(F, G)$ and the counit $FG \rightarrow 1$ by $ad'(F, G)$. Given two smooth k -schemes $X, Y \in \text{Sm}/k$ and two vector bundles $\mathcal{E}, \mathcal{E}'$ over X resp. Y , we write $\mathcal{E} \times \mathcal{E}'/X \times Y$ for the external sum over $X \times_k Y$.

2. \mathbb{A}^1 -HOMOTOPY CATEGORY

2.1. Unstable \mathbb{A}^1 -homotopy category. Let Sm/k denote the category of separated smooth schemes of finite type over a field k . We write Spc/k for the category $\Delta^{op}Sh_{Nis}(\text{Sm}/k)$ consisting of simplicial Nisnevich sheaves of sets on Sm/k . An object in Spc/k is simply called a k -space, which is usually denoted by calligraphic letter \mathcal{X} . The Yoneda embedding $\text{Sm}/k \rightarrow \text{Spc}/k$ is given by sending a smooth scheme $X \in \text{Sm}/k$ to the corresponding representable sheaf $\text{Hom}_{\text{Sm}/k}(-, X)$ then by taking the associated constant simplicial object, where all face and degeneracy maps are the identity. We will identify Sm/k with its essential image in Spc/k . Denote by Spc_+/k the category of pointed k -space, whose objects are (\mathcal{X}, x) , where \mathcal{X} is a k -space and $x : \text{Spec } k \rightarrow \mathcal{X}$ is a distinguished point. One has an adjoint pair

$$\text{Spc}/k \rightleftarrows \text{Spc}_+/k,$$

which means that the functor $\text{Spc}/k \rightarrow \text{Spc}_+/k$ sending $\mathcal{X} \rightarrow \mathcal{X}_+ = \mathcal{X} \amalg \text{Spec } k$ is left-adjoint to the forgetful functor $\text{Spc}_+/k \rightarrow \text{Spc}/k$. The category Spc/k can be equipped with the injective local model structure (C_s, W_s, F_s) , where cofibrations are monomorphisms, weak

equivalences are stalkwise weak equivalences of simplicial sets and fibrations are morphisms with right lifting property wrt. morphisms in $C_s \cap W_s$. Denote by $\mathbf{Ho}_s^{Nis}(k)$ the resulting unpointed homotopy category as constructed by Joyal-Jardine (cf. [MV01, §2 Thm. 1.4]). We will write $\mathbf{Ho}_{s,+}^{Nis}(k)$ for the pointed homotopy category.

Definition 2.1. [MV01]

- (1) A k -space $\mathcal{Z} \in Spc/k$ is called \mathbb{A}^1 -local if and only for any object $\mathcal{X} \in Spc/k$, the projection $\mathcal{X} \times \mathbb{A}^1 \rightarrow \mathcal{X}$ induces a bijection

$$\mathrm{Hom}_{\mathbf{Ho}_s^{Nis}(k)}(\mathcal{X}, \mathcal{Z}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Ho}_s^{Nis}(k)}(\mathcal{X} \times \mathbb{A}^1, \mathcal{Z}).$$

- (2) Let $\mathcal{X} \rightarrow \mathcal{Y} \in \mathrm{Mor}(Spc/k)$ be a morphism of k -spaces. It is an \mathbb{A}^1 -weak equivalence if and only for any \mathbb{A}^1 -local object \mathcal{Z} , the induced map

$$\mathrm{Hom}_{\mathbf{Ho}_s^{Nis}(k)}(\mathcal{Y}, \mathcal{Z}) \rightarrow \mathrm{Hom}_{\mathbf{Ho}_s^{Nis}(k)}(\mathcal{X}, \mathcal{Z})$$

is bijective.

In [MV01, §2 Thm. 3.2], F. Morel and V. Voevodsky proved that Spc/k can be endowed with the \mathbb{A}^1 -local injective model structure $(C, W_{\mathbb{A}^1}, F_{\mathbb{A}^1})$, where cofibrations are monomorphisms, weak equivalences are \mathbb{A}^1 -weak equivalences. The associated homotopy category obtained from Spc/k by inverting \mathbb{A}^1 -weak equivalences is denoted by $\mathbf{Ho}_{\mathbb{A}^1}(k) \stackrel{def}{=} Spc/k[W_{\mathbb{A}^1}^{-1}]$. This category is called the unstable \mathbb{A}^1 -homotopy category of smooth k -schemes. Let $\mathbf{Ho}_{s,\mathbb{A}^1-loc}^{Nis}(k) \subset \mathbf{Ho}_s^{Nis}(k)$ be the full subcategory consisting of \mathbb{A}^1 -local objects. In fact, one has an adjoint pair (cf. [MV01])

$$L_{\mathbb{A}^1} : \mathbf{Ho}_s^{Nis}(k) \rightleftarrows \mathbf{Ho}_{s,\mathbb{A}^1-loc}^{Nis}(k) : i,$$

where $L_{\mathbb{A}^1}$ is the \mathbb{A}^1 -localization functor sending \mathbb{A}^1 -weak equivalences to isomorphisms. $L_{\mathbb{A}^1}$ induces thus an equivalence of categories $\mathbf{Ho}_{\mathbb{A}^1}(k) \rightarrow \mathbf{Ho}_{s,\mathbb{A}^1-loc}^{Nis}(k)$. This will imply that if $\mathcal{X} \in Spc/k$ is any object and \mathcal{Y} is an \mathbb{A}^1 -local object, then one has a canonical bijection

$$\mathrm{Hom}_{\mathbf{Ho}_s^{Nis}(k)}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Ho}_{\mathbb{A}^1}(k)}(\mathcal{X}, \mathcal{Y}).$$

We will write $\mathbf{Ho}_{\mathbb{A}^1,+}(k)$ for the unstable pointed \mathbb{A}^1 -homotopy category of smooth k -schemes. Recall

Definition 2.2. Let $X \in Sm/k$ and E be a vector bundle over X . The Thom space of E is the pointed sheaf

$$Th(E/X) = E/E - s_0(X),$$

where $s_0 : X \rightarrow E$ is the zero section of E .

Let $T \in Spc_+/k$ be the quotient sheaf $\mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ pointed by the image of $\mathbb{A}^1 - \{0\}$. Then $T \cong S_t^1 \wedge S_s^1$ in $\mathbf{Ho}_{\mathbb{A}^1,+}(k)$ ([MV01, Lem. 2. 15]). For a pointed space $\mathcal{X} \in Spc_+/k$, we denote by $\Sigma_T(\mathcal{X}, x) = T \wedge (\mathcal{X}, x)$. Remark that $\mathbb{P}^n/\mathbb{P}^{n-1} \cong T^n \stackrel{def}{=} T^{\wedge n}$ is an \mathbb{A}^1 -equivalence. In particular, we have $(\mathbb{P}^1, *) \cong T$ ([MV01, Cor. 2.18]). Recall

Proposition 2.3. [MV01, §3 Prop. 2. 17] Let $X, Y \in Sm/k$ and E, E' be vector bundles on X and Y respectively. One has

- (1) There is a canonical isomorphism of pointed sheaves

$$Th(E \times E'/X \times Y) = Th(E/X) \wedge Th(E'/Y).$$

(2) *There is a canonical isomorphism of pointed sheaves*

$$Th(\mathcal{O}_X^n) = \Sigma_T^n X_+$$

(3) *The canonical morphism of pointed sheaves*

$$\mathbb{P}(E \oplus \mathcal{O}_X)/\mathbb{P}(E) \rightarrow Th(E)$$

is an \mathbb{A}^1 -weak equivalence.

The following theorem due to Voevodsky will play an essential role for our purpose.

Theorem 2.4. [Voe03, Thm. 2.11] *Let $X \in SmProj/k$ a smooth projective variety of pure dimension d_X over a field k . There exists an integer n_X and a vector bundle V_X over X of rank n_X , such that*

$$V_X \oplus T_X = \mathcal{O}_X^{n_X + d_X} \in K_0(X),$$

where T_X denotes the tangent bundle of X . Moreover, there exists a morphism $T_X^{n_X + d_X} \rightarrow Th(V_X)$ in $\mathbf{Ho}_{\mathbb{A}^1,+}(k)$, such that the induced map $H_{\mathcal{M}}^{2d_X}(X, \mathbb{Z}(d_X)) \rightarrow \mathbb{Z}$ coincides with the degree map $\deg : CH_0(X) \rightarrow \mathbb{Z}$.

2.2. Stable \mathbb{A}^1 -homotopy categories. Let $\mathbf{Spect}^\Sigma(Spc/k)$ be the category of symmetric spectra in k -spaces, which can be viewed as category of Nisnevich sheaves of symmetric spectra. By applying the construction in [Ay08, Def. 4.4.40, Cor. 4.4.42, Prop. 4.4.62], $\mathbf{Spect}^\Sigma(Spc/k)$ has the structure of a monoidal model category. Let $\mathbf{StHo}_{S^1}(k)$ be the resulting homotopy category. The stable \mathbb{A}^1 -homotopy category of S^1 -spectra $\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)$ is obtained from $\mathbf{StHo}_{S^1}(k)$ by Bousfield localization. Equivalently, the category $\mathbf{Spect}^\Sigma(Spc/k)$ can be equipped with an \mathbb{A}^1 -local model structure (cf. [Ay08, Def. 4.5.12]). The homotopy category of this \mathbb{A}^1 -local model structure is $\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)$, which is also known to be equivalent to the category $\mathbf{StHo}_{\mathbb{A}^1-loc}^{S^1}(Sm/k)$ constructed by F. Morel in [Mor05, Def. 4.1.1]. The \mathbb{A}^1 -local symmetric sphere spectrum is defined by taking the functor

$$\underline{n} \mapsto L_{\mathbb{A}^1}(S_s^{1 \wedge n})$$

with an action of symmetric groups, where $L_{\mathbb{A}^1}$ denotes the \mathbb{A}^1 -localization functor. For a pointed space (\mathcal{X}, x) , its \mathbb{A}^1 -local symmetric suspension spectrum is defined as the symmetric sequence

$$\underline{n} \mapsto L_{\mathbb{A}^1}(S_s^{1 \wedge n} \wedge \mathcal{X})$$

together with symmetric groups actions. Let \mathcal{E} be an \mathbb{A}^1 -local symmetric spectrum in Spc/k . One defines ([AH11, Def. 2.1.11]) the i -th S^1 -stable \mathbb{A}^1 -homotopy sheaf $\pi_i^{st\mathbb{A}^1, S^1}(\mathcal{E})$ of \mathcal{E} as the Nisnevich sheaf on Sm/k associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)}(S_s^{1 \wedge i} \wedge \Sigma_s^\infty U_+, \mathcal{E}).$$

Now we consider the symmetric T -spectra or \mathbb{P}^1 -spectra. \mathbb{P}^1 is pointed with ∞ and $\mathbb{P}^{1 \wedge n}$ has a natural action of Σ_n by permutation of the factors, so the association $\underline{n} \mapsto \mathbb{P}^{1 \wedge n}$ is a symmetric sequence. A symmetric \mathbb{P}^1 -spectrum is a symmetric sequence with a module structure over the sphere spectrum \mathbf{S}^0 . Denote by $\mathbf{Spect}_{\mathbb{P}^1}^\Sigma(Spc/k)$ the full subcategory of the category of symmetric sequence in k -spaces $\mathbf{Fun}(\mathbf{Sym}, Spc_+/k)$ consisting of symmetric \mathbb{P}^1 -spectra, which also has a model structure [Ay08, Def. 4.5.21]. Here we denote by \mathbf{Sym} the groupoid, whose objects are \underline{n} and morphisms are given by bijections. Let $\mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)$ be the resulting homotopy category, which is called \mathbb{P}^1 -stable \mathbb{A}^1 -homotopy category. For

a pointed space (\mathcal{X}, x) , we will write $\Sigma_{\mathbb{P}^1}^\infty(\mathcal{X}, x)$ for the suspension symmetric \mathbb{P}^1 -spectrum, i.e., it is given by the functor $\underline{n} \mapsto \mathbb{P}^{1 \wedge n} \wedge \mathcal{X}$ equipped with an action of symmetric group by permuting the first n -factors. Let \mathbf{S}^i be a suspension symmetric \mathbb{P}^1 -spectrum of S_s^i . If \mathcal{E} is a symmetric \mathbb{P}^1 -spectrum, then the i -th \mathbb{P}^1 -stable \mathbb{A}^1 -homotopy sheaf $\pi_i^{st\mathbb{A}^1, \mathbb{P}^1}(\mathcal{E})$ is defined as the Nisnevich sheaf on Sm/k associated to the presheaf (cf. [AH11, Def. 2.1.14])

$$U \mapsto \mathrm{Hom}_{\mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)}(\mathbf{S}^i \wedge \Sigma_{\mathbb{P}^1}^\infty U_+, \mathcal{E}).$$

Theorem 2.5. [Mor05, Thm. 6.1.8 and Cor. 6.2.9] *Let \mathcal{E} be an \mathbb{A}^1 -local symmetric S^1 -spectrum. The homotopy sheaves $\pi_i^{st\mathbb{A}^1, S^1}(\mathcal{E})$ are strictly \mathbb{A}^1 -invariant.*

One has a canonical isomorphism [AH11, Prop. 2.1.16]

$$\begin{aligned} \mathrm{colim}_n \mathrm{Hom}_{\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)}(\Sigma_s^\infty \mathbb{G}_m^{\wedge n} \wedge \Sigma_s^\infty(U_+), \Sigma_s^\infty \mathbb{G}_m^{\wedge n} \wedge \Sigma_s^\infty(\mathcal{X}, x)) &\xrightarrow{\cong} \\ \mathrm{Hom}_{\mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)}(\Sigma_{\mathbb{P}^1}^\infty(U_+), \Sigma_{\mathbb{P}^1}^\infty(\mathcal{X}, x)). \end{aligned}$$

So one may view that $\mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)$ is obtained from $\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)$ by formally inverting the \mathbb{A}^1 -localized suspension spectrum of \mathbb{G}_m . So from 2.5, we see that for a pointed k -space (\mathcal{X}, x) , the homotopy sheaves $\pi_i^{st\mathbb{A}^1, \mathbb{P}^1}(\mathcal{X})$ are also strictly \mathbb{A}^1 -invariant. By the computation of F. Morel, one can identify the Milnor-Witt K -theory sheaves with stable homotopy sheaves of spheres

$$\mathbf{K}_n^{MW} \stackrel{def}{=} \pi_0^{st\mathbb{A}^1, \mathbb{P}^1}(\Sigma_{\mathbb{P}^1}^\infty(\mathbb{G}_m^{\wedge n})).$$

This identification allows us to conclude that \mathbf{K}_n^{MW} are strictly \mathbb{A}^1 -invariant sheaves.

3. \mathbb{A}^1 -HOMOLOGICAL ALGEBRA

3.1. Effective \mathbb{A}^1 -derived category. Let $Ch_-(\mathcal{A}b_k)$ be the category of chain complexes over the category $\mathcal{A}b_k$ of abelian Nisnevich sheaves. Denote by $Ch_{\geq 0}(\mathcal{A}b_k)$ the category of chain complexes of abelian Nisnevich sheaves, whose homological degree ≥ 0 . The sheaf-theoretical Dold-Kan correspondence

$$N : \Delta^{op} \mathcal{A}b_k \rightleftarrows Ch_{\geq 0}(\mathcal{A}b_k) : K,$$

where $\Delta^{op} \mathcal{A}b_k$ is the category of simplicial abelian Nisnevich sheaves, gives us via the inclusion functor $Ch_{\geq 0}(\mathcal{A}b_k) \hookrightarrow Ch_-(\mathcal{A}b_k)$, a functor

$$\Delta^{op}(\mathcal{A}b_k) \rightarrow Ch_-(\mathcal{A}b_k).$$

By applying this functor on the Eilenberg-MacLane spectrum $H\mathbb{Z}$, we obtain a ring spectrum $\widetilde{H\mathbb{Z}}$ in $\mathbf{Fun}(Sym, Ch_-(\mathcal{A}b_k))$. Let $\mathbf{Spect}^\Sigma(Ch_-(\mathcal{A}b_k))$ be the full subcategory of the category $\mathbf{Fun}(Sym, Ch_-(\mathcal{A}b_k))$ consisting of modules over $\widetilde{H\mathbb{Z}}$. On the other hand, by composing with the free abelian group functor

$$\mathbb{Z}(-) : Spc/k \rightarrow \Delta^{op}(\mathcal{A}b_k),$$

one obtains a functor

$$\mathbf{Fun}(Sym, Spc_+/k) \rightarrow \mathbf{Fun}(Sym, Ch_-(\mathcal{A}b_k)),$$

which sends the sphere symmetric sequence to $\widetilde{H\mathbb{Z}}$. This induces then a functor between categories of symmetric spectra

$$\mathbf{Spect}^\Sigma(Spc/k) \rightarrow \mathbf{Spect}^\Sigma(Ch_-(\mathcal{A}b_k)).$$

In fact, by [Hov01, Thm. 9.3], this induces a Quillen functor, which one refers as Hurewicz functor

$$\mathfrak{H}^{ab} : \mathbf{StHo}_{S^1}(k) \rightarrow D_-(\mathcal{A}b_k).$$

Now the effective \mathbb{A}^1 -derived category $D_{\mathbb{A}^1}^{eff}(k)$ is constructed by applying \mathbb{A}^1 -localization on the category $\mathbf{Spect}^{\Sigma}(Ch_-(\mathcal{A}b_k))$. By the work of Cisinski and Déglise (cf. [CD10, §5]), this category is equivalent to the \mathbb{A}^1 -derived category constructed by F. Morel in [Mor12]. Let $(\mathcal{X}, x) \in Spc_+/k$ be a pointed space, and $\Sigma_s^{\infty}(\mathcal{X}, x)$ its suspension symmetric spectrum. We apply the Hurewicz functor on $\Sigma_s^{\infty}(\mathcal{X}, x)$ and then $L_{\mathbb{A}^1}^{ab}(-)$, so we may define a functor

$$\tilde{C}_*^{\mathbb{A}^1} : \mathbf{StHo}_{S^1}(k) \rightarrow D_{\mathbb{A}^1}^{eff}(k), \quad \Sigma_s^{\infty}(\mathcal{X}, x) \mapsto L_{\mathbb{A}^1}^{ab}(\mathfrak{H}^{ab}(\Sigma_s^{\infty}(\mathcal{X}, x))).$$

Here we write $L_{\mathbb{A}^1}^{ab}$ for the \mathbb{A}^1 -localization functor on chain complexes to distinguish from the \mathbb{A}^1 -localization $L_{\mathbb{A}^1}$ on spaces. If $\mathcal{X} \in Spc/k$ is not pointed, then we write $C_*^{\mathbb{A}^1}(\mathcal{X}) \stackrel{def}{=} \tilde{C}_*^{\mathbb{A}^1}(\mathcal{X}_+)$. Define $\mathbb{Z}[n] = \mathfrak{H}^{ab}(\Sigma_s^{\infty}S^n)$.

Definition 3.1. *Let $\mathcal{X} \in Spc/k$ be a k -space. Its i -th \mathbb{A}^1 -homology sheaf is the Nisnevich sheaf $\mathbf{H}_i^{\mathbb{A}^1}(\mathcal{X})$ associated to the presheaf*

$$U \mapsto \mathrm{Hom}_{D_{\mathbb{A}^1}^{eff}(k)}(C_*^{\mathbb{A}^1}(U)[i], C_*^{\mathbb{A}^1}(\mathcal{X})) \stackrel{def}{=} \mathrm{Hom}_{D_{\mathbb{A}^1}^{eff}(k)}(C_*^{\mathbb{A}^1}(U) \otimes \mathbb{Z}[i], C_*^{\mathbb{A}^1}(\mathcal{X})).$$

Consider (\mathbb{P}^1, ∞) pointed by ∞ . According to [MV01, Cor. 2.18], we have $\mathbb{P}^1 = S_s^1 \wedge \mathbb{G}_m$, so we have an identification $\tilde{C}_*^{\mathbb{A}^1}(\mathbb{P}^1) = \tilde{C}_*^{\mathbb{A}^1}(S_s^1 \wedge \mathbb{G}_m)$. We define the \mathbb{A}^1 -Tate complex (called enhanced Tate (motivic) complex by A. Asok and C. Haesemeyer [AH11, Def. 2.1.25 and Def. 3.2.1 and Lem. 3.2.2]) as

$$\mathbb{Z}_{\mathbb{A}^1}(n) \stackrel{def}{=} \tilde{C}_*^{\mathbb{A}^1}(\mathbb{P}^{1 \wedge n})[-2n] = \mathbb{Z}_{\mathbb{A}^1}(1)^{\otimes n}.$$

Definition 3.2. *Let $\mathcal{X} \in Spc/k$ be a k -space. The bigraded unstable \mathbb{A}^1 -cohomology group $H_{\mathbb{A}^1}^{p,q}(\mathcal{X}, \mathbb{Z})$ is defined as*

$$H_{\mathbb{A}^1}^{p,q}(\mathcal{X}, \mathbb{Z}) = \mathrm{Hom}_{D_{\mathbb{A}^1}^{eff}(k)}(C_*^{\mathbb{A}^1}(\mathcal{X}), \mathbb{Z}_{\mathbb{A}^1}(q)[p]).$$

The relationship between unstable \mathbb{A}^1 -cohomology and Nisnevich hypercohomology with coefficient $\mathbb{Z}_{\mathbb{A}^1}(n)$ is given by the following

Proposition 3.3. [AH11, Prop. 3.2.5] *Let k be a field and $\mathcal{X} \in Spc/k$ be a k -space. One has*

- (1) *For any p, q , there is a canonical isomorphism*

$$\mathbb{H}_{Nis}^p(\mathcal{X}, \mathbb{Z}_{\mathbb{A}^1}(q)) \xrightarrow{\sim} H_{\mathbb{A}^1}^{p,q}(\mathcal{X}, \mathbb{Z}).$$

- (2) *The cohomology sheaves $\underline{H}^p(\mathbb{Z}_{\mathbb{A}^1}(q)) = 0$, if $p > q$.*
(3) *There is a canonical isomorphism $\underline{H}^p(\mathbb{Z}_{\mathbb{A}^1}(p)) \cong \mathbf{K}_p^{MW}$, for all $p > 0$.*

Remark 3.4. One observes that the complex $\mathbb{Z}_{\mathbb{A}^1}(n)$ is \mathbb{A}^1 -local, hence by definition (cf. [Mor12, Def. 5.17]) one has immediately that the sheaves $\underline{H}^p(\mathbb{Z}_{\mathbb{A}^1}(q))$ are strictly \mathbb{A}^1 -invariant.

3.2. \mathbb{P}^1 -stable \mathbb{A}^1 -derived category. Having \mathbb{A}^1 -Tate complex, the way that we stabilize the category $D_{\mathbb{A}^1}^{eff}(k)$ is to invert formally the \mathbb{A}^1 -Tate complex to obtain the \mathbb{P}^1 -stable \mathbb{A}^1 -derived category $D_{\mathbb{A}^1}(k)$. This can be done by [CD10, §5]. As before, we take $D_{\mathbb{A}^1}(k)$ as the resulting homotopy category of the model category $\mathbf{Spect}_{\mathbb{P}^1}^{\Sigma}(Ch_-(\mathcal{A}b_k))$ consisting of modules over the \mathbb{A}^1 -localization of the normalized chain complex of the free abelian group on the sphere symmetric \mathbb{P}^1 -spectrum. For a pointed space $(\mathcal{X}, x) \in Spc_+/k$, the stable \mathbb{A}^1 -complex $\tilde{C}_*^{st\mathbb{A}^1}(\mathcal{X})$ of (\mathcal{X}, x) is defined as $L_{\mathbb{A}^1}^{ab}(N\mathbb{Z}(\Sigma_{\mathbb{P}^1}^{\infty}(\mathcal{X}, x)))$ and if $\mathcal{X} \in Spc/k$ is an unpointed k -space, then we write $C_*^{st\mathbb{A}^1}(\mathcal{X})$ for $\tilde{C}_*^{st\mathbb{A}^1}(\mathcal{X}_+)$. The category $D_{\mathbb{A}^1}(k)$ has an unit object, denoted by $\mathbf{1}_k$, which is the complex $\tilde{C}_*^{st\mathbb{A}^1}(\mathbf{S}^0)$. Define $\mathbf{1}_k[n] = \mathbf{1}_k \otimes \tilde{C}_*^{st\mathbb{A}^1}(S_s^n)$ and $\tilde{C}_*^{st\mathbb{A}^1}(\mathcal{X})[n] = \tilde{C}_*^{st\mathbb{A}^1}(\mathcal{X}) \otimes \mathbf{1}_k[n]$ for a k -space $(\mathcal{X}, x) \in Spc_+/k$.

Definition 3.5. *Let $\mathcal{X} \in Spc/k$ be a k -space. The i -th \mathbb{P}^1 -stable \mathbb{A}^1 -homology sheaf $\mathbf{H}_i^{st\mathbb{A}^1}(\mathcal{X})$ is the Nisnevich sheaf associated to the presheaf*

$$U \mapsto \mathrm{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{st\mathbb{A}^1}(U)[i], C_*^{st\mathbb{A}^1}(\mathcal{X})).$$

Just like in case of stable \mathbb{A}^1 -homotopy categories, one has the following result

Proposition 3.6. [AH11, Prop. 2.1.29] *Let $U \in Sm/k$ and $(\mathcal{X}, x) \in Spc_+/k$. One has a canonical isomorphism*

$$(3.1) \quad \mathrm{colim}_n \mathrm{Hom}_{D_{\mathbb{A}^1}^{eff}(k)}(C_*^{\mathbb{A}^1}(U) \otimes \mathbb{Z}_{\mathbb{A}^1}(n)[i], \tilde{C}_*^{\mathbb{A}^1}(\mathcal{X}) \otimes \mathbb{Z}_{\mathbb{A}^1}(n)[i]) \xrightarrow{\cong} \mathrm{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{st\mathbb{A}^1}(U), \tilde{C}_*^{st\mathbb{A}^1}(\mathcal{X})).$$

The Hurewicz formalism induces the following functors, which one still calls Hurewicz functors (or abelianization functors)

$$\begin{aligned} \mathbf{StHo}_{\mathbb{A}^1, S^1}(k) &\rightarrow D_{\mathbb{A}^1}^{eff}(k), \\ \mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k) &\rightarrow D_{\mathbb{A}^1}(k), \end{aligned}$$

which give rise to morphisms of sheaves

$$\begin{aligned} \pi_i^{st\mathbb{A}^1, S^1}(\Sigma_s^{\infty}(\mathcal{X}_+)) &\rightarrow \mathbf{H}_i^{\mathbb{A}^1}(\mathcal{X}), \\ \pi_i^{st\mathbb{A}^1, \mathbb{P}^1}(\Sigma_{\mathbb{P}^1}^{\infty}(\mathcal{X}_+)) &\rightarrow \mathbf{H}_i^{st\mathbb{A}^1}(\mathcal{X}). \end{aligned}$$

Definition 3.7. *Let $\mathcal{X} \in Spc/k$ be a k -space. The bigraded \mathbb{P}^1 -stable \mathbb{A}^1 -cohomology group $H_{st\mathbb{A}^1}^{p,q}(\mathcal{X}, \mathbb{Z})$ is defined as*

$$H_{st\mathbb{A}^1}^{p,q}(\mathcal{X}, \mathbb{Z}) = \mathrm{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{st\mathbb{A}^1}(\mathcal{X}), \mathbb{Z}_{\mathbb{A}^1}(q)[p]).$$

The advantage of \mathbb{P}^1 -stable \mathbb{A}^1 -derived category $D_{\mathbb{A}^1}(k)$ is that one has duality formalism. In the context of stable \mathbb{A}^1 -homotopy theory, it was done in [Hu05, App. A]. The theorem of Voevodsky 2.4 implies that one has a morphism

$$\mathbf{1}_k \rightarrow \Sigma_{\mathbb{P}^1}^{\infty} Th(-T_X).$$

We recall

Proposition 3.8. [AH11, Prop. 3.5.2 and Lem. 3.5.3] *Let $X \in \text{SmProj}/k$, then $C_*^{st\mathbb{A}^1}(X)$ is a strong dualizable object in $D_{\mathbb{A}^1}(k)$ and its dual is $C_*^{st\mathbb{A}^1}(X)^\vee = \tilde{C}_*^{st\mathbb{A}^1}(\text{Th}(-T_X))$. Consequently, one has a canonical isomorphism*

$$(3.2) \quad \text{Hom}_{D_{\mathbb{A}^1}(k)}(\mathbf{1}_k, C_*^{st\mathbb{A}^1}(X)) \xrightarrow{\cong} \text{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{st\mathbb{A}^1}(X)^\vee, \mathbf{1}_k).$$

We end up this section by a definition

Definition 3.9. *Let k be a field. One defines the geometric stable \mathbb{A}^1 -derived category $D_{\mathbb{A}^1, gm}(k)$ over k as the full subcategory of $D_{\mathbb{A}^1}(k)$ generated by $C_*^{st\mathbb{A}^1}(X)$ for $X \in \text{SmProj}/k$.*

4. TWISTED CHOW-WITT CORRESPONDENCES

We recall the intersection theory in Chow-Witt rings in [Fas07] and [Fas08]. Let k be a field of characteristic different from 2, $X \in \text{Sm}/k$ be a smooth k -scheme and \mathcal{L} a line bundle on X . The i -th Chow-Witt complex twisted by \mathcal{L} is defined to be the fiber product complex [Fas07, Def. 3.21]

$$\begin{array}{ccc} C^*(X, G^i, \mathcal{L}) & \longrightarrow & C_{FS}^*(X, I^i, \mathcal{L}) \\ \downarrow & & \downarrow \\ C^*(X, K_i^M) & \longrightarrow & C_{FS}^*(X, I^i/I^{i+1}) \end{array}$$

This means that one has a short exact sequence of complexes

$$0 \rightarrow C^*(X, G^i, \mathcal{L}) \rightarrow C_{FS}^*(X, I^i, \mathcal{L}) \oplus C^*(X, K_i^M) \rightarrow C_{FS}^*(X, I^i/I^{i+1}) \rightarrow 0.$$

Here $C^*(X, K_i^M)$ is the Gersten-Milnor complex

$$0 \rightarrow K_i^M(k(X)) \rightarrow \bigoplus_{x \in X^{(1)}} K_{i-1}^M(\kappa(x)) \rightarrow \cdots$$

$C_{FS}^*(X, I^i, \mathcal{L})$ is the twisted by \mathcal{L} complex of fundamental ideal of the Witt ring of modules of finite length in Fasel's notation. The i -th twisted by \mathcal{L} Chow-Witt group $\widetilde{\text{CH}}^i(X, \mathcal{L})$ is defined as $H^i(X, C^*(X, G^i, \mathcal{L}))$. There is a product (cf. [Fas07, Rem. 6.2])

$$\cdot : H^i(C^*(X, G^i, \mathcal{L})) \otimes H^j(C^*(X, G^j, \mathcal{L}')) \rightarrow H^{i+j}(C^*(X, G^{i+j}, \mathcal{L} \otimes \mathcal{L}')),$$

which is not commutative in general. This product defines then the intersection product \cdot on the twisted Chow-Witt groups $\text{CH}^*(X, \mathcal{L})$. For $\alpha \in \text{CH}^i(X, \mathcal{L})$ and $\beta \in \text{CH}^j(X, \mathcal{L}')$, we define their left resp. right intersection product as

$$\begin{aligned} \alpha \cdot_l \beta &= \alpha \cdot \beta, \\ \alpha \cdot_r \beta &= \beta \cdot \alpha. \end{aligned}$$

Recall the following proper base change result of J. Fasel:

Lemma 4.1. [AH11, Prop. 4.2.2] *Given a cartesian square \square of smooth k -schemes*

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{p} & Y \end{array}$$

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where f is proper. Then via the exchange transformation structure

$$Ex(\square_*^!) : p^! f_* \xrightarrow{ad(g^!, g_*)} g_* g^! p^! f_* = g_* q^! f^! f_* \xrightarrow{ad'(f^!, f_*)} g_* q^!$$

one has $p^! f_* = g_* q^!$.

Remark 4.2. (1) $p^! = p^*$ and $q^! = q^*$, if they are flat (see [Fas07, Prop. 7.4]).

(2) The formulation of 4.1 with exchange transformation structure is not exactly what is proved in [AH11, Prop. 4.2.2]. However, as remarked by B. Calmès and J. Hornbostel [CH09], that the adjunction formalism on categorical level should apply also to Chow-Witt theory.

(3) J. Fasel (private communication) pointed out to me that Lemma 4.1 is not quite correct in general, since in general one has an excess intersection formula. However, the projection formula holds for Chow-Witt groups and that is what we need in this work.

Corollary 4.3. [AH11, Cor. 4.2.3 and Rem. 4.2.4] *Let $f : X \rightarrow Y$ be a proper morphism of smooth k -schemes. Then one has projection formulas for both left and right intersection products with arbitrary twists*

$$f_*(f^! \alpha \cdot_l \beta) = \alpha \cdot_l f_* \beta,$$

and

$$f_*(f^! \alpha \cdot_r \beta) = \alpha \cdot_r f_* \beta.$$

Let $X, Y \in SmProj/k$. We define the group of Chow-Witt correspondence between X and Y as $\widetilde{CH}^{d_X}(X \times Y, p_X^{XY*} \omega_{X/k})$, where p_X^{XY} denotes the projection $X \times Y \rightarrow X$, $\omega_{X/k}$ is the canonical line bundle on X and $d_X = \dim(X)$.

Proposition 4.4. *Let $X, Y, Z \in SmProj/k$, where k is a field of $\text{char}(k) \neq 2$. If $\alpha \in \widetilde{CH}^{d_X}(X \times Y, p_X^{XY*} \omega_{X/k})$ and $\beta \in \widetilde{CH}^{d_Y}(Y \times Z, p_Y^{YZ*} \omega_{Y/k})$, then their composition will land in*

$$\beta \circ \alpha \stackrel{\text{def}}{=} p_{XZ*}^{XYZ}(p_{XY}^{XYZ*}(\alpha) \cdot_l p_{YZ}^{XYZ*}(\beta)) \in \widetilde{CH}^{d_X}(X \times Z, p_X^{XZ*} \omega_{X/k})$$

Moreover, this composition is associative and satisfies the unit axiom.

Proof. The projection $p_{XY}^{XYZ} : X \times Y \times Z \rightarrow X \times Y$ induces a functorial homomorphism (cf. [Fas07, Thm. 3.27] or [Fas08, Chap. 10])

$$p_{XY}^{XYZ*} : \widetilde{CH}^{d_X}(X \times Y, p_X^{XY*} \omega_{X/k}) \rightarrow \widetilde{CH}^{d_X}(X \times Y \times Z, p_{XY}^{XYZ*} \circ p_X^{XY*} \omega_{X/k}).$$

The right hand-side is nothing but $\widetilde{CH}^{d_X}(X \times Y \times Z, p_X^{XYZ*} \omega_{X/k})$. Analogously, we have a functorial homomorphism on $Y \times Z$ factor. Consider the intersection product defined as pullback of the exterior product along the diagonal (cf. [Fas07, Def. 6.1])

$$\begin{aligned} \cdot : \widetilde{CH}^{d_X}(X \times Y \times Z, p_X^{XYZ*} \omega_{X/k}) \otimes \widetilde{CH}^{d_Y}(X \times Y \times Z, p_Y^{XYZ*} \omega_{Y/k}) \rightarrow \\ \widetilde{CH}^{d_X+d_Y}(X \times Y \times Z, p_X^{XYZ*} \omega_{X/k} \otimes_{\mathcal{O}_{X \times Y \times Z}} p_Y^{XYZ*} \omega_{Y/k}). \end{aligned}$$

In general, the intersection product defined on $H^j(C^*(X, G^j, \mathcal{L}))$ is not commutative (not even anti-commutative), see [Fas07, Rem. 6.7]. So we define the composition $\beta \circ \alpha$ via the left intersection product.

$$\beta \circ \alpha \stackrel{\text{def}}{=} p_{XZ*}^{XYZ}(p_{XY}^{XYZ*}(\alpha) \cdot_l p_{YZ}^{XYZ*}(\beta))$$

Since

$$\omega_{X \times Y \times Z/k} = p_X^{XYZ*} \omega_{X/k} \otimes p_Y^{XYZ*} \omega_{Y/k} \otimes p_Z^{XYZ*} \omega_{Z/k},$$

we may rewrite

$$\begin{aligned} \widetilde{\text{CH}}^{d_X+d_Y}(X \times Y \times Z, p_X^{XYZ*} \omega_{X/k} \otimes_{\mathcal{O}_{X \times Y \times Z}} p_Y^{XYZ*} \omega_{Y/k}) = \\ \widetilde{\text{CH}}^{d_X+d_Y}(X \times Y \times Z, \omega_{X \times Y \times Z} \otimes (p_Z^{XYZ*} \omega_{Z/k})^\vee). \end{aligned}$$

But $(p_Z^{XYZ*} \omega_{Z/k})^\vee \cong p_Z^{XYZ*}(\omega_{Z/k}^\vee)$. Apply now the pushforward p_{XZ*}^{XYZ} , we have

$$\begin{aligned} p_{XZ*}^{XYZ} : \widetilde{\text{CH}}^{d_X+d_Y}(X \times Y \times Z, \omega_{X \times Y \times Z/k} \otimes p_{XZ}^{XYZ*} \circ p_Z^{XZ*}(\omega_{Z/k}^\vee)) \rightarrow \\ \widetilde{\text{CH}}^{d_X}(X \times Z, \omega_{X \times Z} \otimes p_Z^{XZ*}(\omega_{Z/k}^\vee)) = \widetilde{\text{CH}}^{d_X}(X \times Z, p_X^{XZ*} \omega_{X/k}), \end{aligned}$$

which gives us the composition $\beta \circ \alpha$ as expected. Now we check the unit axiom. Given $X, Y \in \text{SmProj}/k$ and $\alpha \in \widetilde{\text{CH}}^{d_X}(X \times Y, p_X^{XY*} \omega_{X/k})$. Apply the definition of composition we have

$$\Delta_Y \circ \alpha = p_{XY*}^{YYY}(p_{XY}^{YYY*}(\alpha) \cdot p_{YY}^{YYY*}(\Delta_Y)).$$

By projection formula for left intersection product 4.3, the right hand side is nothing but $\alpha \cdot p_{XY*}^{YYY}(p_{YY}^{YYY*}(\Delta_Y))$. We write $p_{YY}^{YYY} = \text{id}_X \times p_Y^{YY}$, so we have then

$$p_{XY*}^{YYY}(p_{YY}^{YYY*}(\Delta_Y)) = p_{XY*}^{YYY}(1_X \times p_Y^{YY*} \Delta_Y) = 1_{X \times Y} \in \widetilde{\text{CH}}^0(X \times Y) \subset \text{GW}(k(X \times Y)),$$

where $\text{GW}(k(X \times Y))$ denotes the Grothendieck-Witt ring. It follows that $\Delta_Y \circ \alpha = \alpha \cdot 1_{X \times Y}$. Symmetrically, we have $\alpha \circ \Delta_X = 1_{X \times Y} \cdot \alpha$. Now the unit axiom follows from the fact that $1_{X \times Y}$ is a left and right unit ([Fas07, Prop. 6.8]). To check the associativity of the composition, one translates word by word from [Ful98, Prop. 16.1.1], where one has to use 4.1, functorialities of pullback and pushforward and the fact that the intersection product on twisted Chow-Witt groups is associative [Fas07, Prop. 6.6]. Indeed, given $X, Y, Z, W \in \text{SmProj}/k$ and α, β, γ correspondences from X to Y resp. Y to Z resp. Z to W , we have then

$$\begin{aligned} \gamma \circ (\beta \circ \alpha) &= p_{XW*}^{XZW}(p_{XZ}^{XZW*}(p_{XZ*}^{XZW}(p_{XY}^{XYZ*} \alpha \cdot p_{YZ}^{XYZ*} \beta)) \cdot p_{ZW}^{XZW*} \gamma) \\ &= p_{XW*}^{XZW}(p_{XZW*}^{XYZ}(p_{XYZ*}^{XYZ}(p_{XY}^{XYZ*} \alpha \cdot p_{YZ}^{XYZ*} \beta)) \cdot p_{ZW}^{XZW*} \gamma) \\ &= p_{XW*}^{XZW}(p_{XZW*}^{XYZ}((p_{XY}^{XZYW*} \alpha \cdot p_{YZ}^{XZYW*} \beta) \cdot p_{XZW*}^{XZYW*} p_{ZW}^{XZW*} \gamma)) \\ &= p_{XW*}^{XZYW}((p_{XY}^{XZYW*} \alpha \cdot p_{YZ}^{XZYW*} \beta) \cdot p_{ZW}^{XZYW*} \gamma) \\ &= p_{XW*}^{XZYW}(p_{XY}^{XZYW*} \alpha \cdot (p_{YZ}^{XZYW*} \beta \cdot p_{ZW}^{XZYW*} \gamma)). \end{aligned}$$

Symmetrically, one can bring $(\gamma \circ \beta) \circ \alpha$ also into this form, which finishes the proof of our proposition. \square

From 4.4 we thus may define the category of Chow-Witt correspondences $\text{CHW}(k)$.

Definition 4.5. *The category of Chow-Witt correspondences $\text{CHW}(k)$ over a field of characteristic unequal 2 is given by $\text{Obj}(\text{CHW}(k)) = \text{Obj}(\text{SmProj}/k)$ and*

$$\text{Hom}_{\text{CHW}(k)}(X, Y) = \widetilde{\text{CH}}^{\dim(X)}(X \times Y, p_X^{XY*} \omega_{X/k}),$$

where composition of morphisms are defined with respect to the left intersection product.

5. PROOF OF 1.3

Let k be an infinite perfect field of characteristic unequal 2. In this section we prove that one has a fully faithful embedding

$$CHW(k)_{\mathbb{Q}}^{op} \rightarrow D_{\mathbb{A}^1, gm}(k)_{\mathbb{Q}},$$

if we assume the conjectural cancellation in the S^1 -stable motivic homotopy category $\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)$ and the conjecture for decomposition of stable \mathbb{A}^1 -cohomology. We will work in this section mainly with \mathbb{Q} -coefficient. Remark that one has the equivalences of categories:

$$\mathbf{StHo}_{\mathbb{A}^1, S^1}(k)_{\mathbb{Q}} \cong D_{\mathbb{A}^1}^{eff}(k)_{\mathbb{Q}}, \quad \mathbf{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)_{\mathbb{Q}} \cong D_{\mathbb{A}^1}(k)_{\mathbb{Q}}.$$

By a theorem of F. Morel (cf. [AH11, Thm. 4.1.8]), one has a quasi-isomorphism of untwisted complexes

$$C^*(-, K_i^{MW}) \simeq C^*(-, K_i^M) \times_{C_{AH}^*(-, I^i/I^{i+1}, \mathcal{O}_X)} C_{AH}^*(-, I^i, \mathcal{O}_X),$$

where for a line bundle \mathcal{L} over a smooth k -scheme $X \in Sm/k$ the complex $C_{AH}^*(-, I^i, \mathcal{L})$ in degree j in Asok-Haesemeyer's notation is given by [AH11, Def. 4.1.4]

$$C_{AH}^j(X, I^i, \mathcal{L}) = \bigoplus_{x \in X^{(j)}} I_{fl}^i(\mathcal{O}_{X,x}, \mathcal{L}_x).$$

Hence we may redefine the twisted i -th Chow-Witt complex of sheaves as the fiber product complex (cf. [AH11, Def. 4.1.9])

$$\begin{array}{ccc} C^*(-, K_i^{MW}, \mathcal{L}) & \longrightarrow & C_{AH}^*(-, I^i, \mathcal{L}) \\ \downarrow & & \downarrow \\ C^*(-, K_i^M) & \longrightarrow & C_{AH}^*(-, I^i/I^{i+1}) \end{array}$$

The only non-zero cohomology sheaf of $C^*(-, K_i^{MW}, \mathcal{L})$ occurs in degree 0 [AH11, Thm. 4.1.10], and so one defines the twisted Milnor-Witt sheaf on a smooth scheme X with a line bundle \mathcal{L} as ([AH11, Def. 4.1.11])

$$\mathbf{K}_i^{MW}(\mathcal{L}) \stackrel{def}{=} H^0 C^*(-, K_i^{MW}, \mathcal{L}).$$

One has obviously a canonical isomorphism from [AH11, Thm. 4.1.10]

$$(5.3) \quad H_{Nis}^i(X, \mathbf{K}_i^{MW}(\mathcal{L})) \cong \widetilde{CH}^i(X, \mathcal{L}).$$

We recall in the following the result on Thom isomorphism:

Theorem 5.1. (*Thom isomorphism*) [AH11, Thm. 4.2.7] *Let $X \in Sm/k$ a smooth k -scheme over a field k of char(k) $\neq 2$ and E/X be a vector bundle of rank r . Then one has the Thom isomorphism*

$$(5.4) \quad H_{Nis}^p(X, \mathbf{K}_q^{MW}(\det E)) \cong H_{Nis}^{p+r}(Th(E), \mathbf{K}_{q+r}^{MW}).$$

We prove firstly

Lemma 5.2. *Let $X, Y \in Sm/k$ be smooth k -schemes, where k is a field. Denote by $d_Y = \dim(Y)$ and n_Y the rank of the vector bundle V_Y/Y (see 2.4). We write $\mathcal{E} = 0_X \times V_Y/X \times Y$. Assume 1.1 and 1.2, we have then a canonical isomorphism*

$$H_{\mathbb{A}^1}^{2(d_Y+n_Y)}(Th(\mathcal{E}), \mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y)) \cong H_{Nis}^{d_Y+n_Y}(Th(\mathcal{E}), \mathbf{K}_{d_Y+n_Y}^{MW})_{\mathbb{Q}}$$

Proof. Consider the hypercohomology spectral sequence

$$E_2^{p,q} = H_{Nis}^p(Th(\mathcal{E}), \underline{H}^q(\mathbb{Q}_{\mathbb{A}^1}(n_Y + d_Y))) \Rightarrow H_{\mathbb{A}^1}^{p+q}(Th(\mathcal{E}), \mathbb{Q}_{\mathbb{A}^1}(n_Y + d_Y)).$$

As $\underline{H}^q(\mathbb{Q}_{\mathbb{A}^1}(n_Y + d_Y)) = 0$ for $q > n_Y + d_Y$, we have a canonical homomorphism

$$H_{\mathbb{A}^1}^{2(d_Y+n_Y)}(Th(\mathcal{E}), \mathbb{Q}(n_Y + d_Y)) \rightarrow H^{d_Y+n_Y}(Th(\mathcal{E}), \mathbf{K}_{n_Y+d_Y}^{MW}),$$

whose kernel and cokernel are built out of the groups $H_{Nis}^{2(d_Y+n_Y)-i}(Th(\mathcal{E}), \underline{H}^i(\mathbb{Q}_{\mathbb{A}^1}(n_Y + d_Y)))$ and $H_{Nis}^{2(n_Y+d_Y)-i+1}(Th(\mathcal{E}), \underline{H}^i(\mathbb{Q}_{\mathbb{A}^1}(n_Y + d_Y)))$ for $i < n_Y + d_Y$. By Nisnevich excision, after shrinking Y , we may assume \mathcal{E} is trivial, hence we have an isomorphism

$$H_{Nis}^{2(d_Y+n_Y)-i}(Th(\mathcal{E}), \underline{H}^i(\mathbb{Q}(n_Y + d_Y))) \simeq H_{Nis}^{2d_Y+n_Y-i}(X \times Y, \underline{H}^i(\mathbb{Q}(n_Y + d_Y)))_{-(n_Y)}.$$

The right hand side can be compute by comparison to the Zariski cohomology of the Gersten resolution (cf. [CTHK97])

$$\begin{aligned} (\underline{H}^i(\mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y))_{-(n_Y)})|_{X \times Y_{Zar}} &\rightarrow \bigoplus_{x \in X \times Y^{(0)}} \underline{H}^i(\mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y))_{-(n_Y)}(\kappa(x)) \rightarrow \cdots \rightarrow \\ &\rightarrow \bigoplus_{x \in X \times Y^{(q)}} \underline{H}^i(\mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y))_{-(n_Y+q)}(\kappa(x)) \rightarrow \cdots \end{aligned}$$

Remark that $\underline{H}^i(\mathbb{Q}_{\mathbb{A}^1}(j))$ are strictly \mathbb{A}^1 -invariant sheaves and hence their contractions are also \mathbb{A}^1 -invariant sheaves (cf. [AH11, Cor. 2.3.6]), so one may apply [AH11, Prop. 4.1.1]. Now for any finitely generated field L/k , we have by [AH11, Prop. 2.3.7]

$$\underline{H}^i(\mathbb{Q}_{\mathbb{A}^1}(n_Y + d_Y))_{-(2d_Y+2n_Y-i)}(L) = H_{\mathbb{A}^1}^i(L_+ \wedge \mathbb{G}_m^{\wedge(2d_Y+2n_Y-i)}, \mathbb{Q}(n_Y + d_Y)).$$

Since we assume the conjecture 1.1, the right hand side is nothing but the stable \mathbb{A}^1 -cohomology $H_{st\mathbb{A}^1}^{2i-2(n_Y+d_Y)}(L_+, \mathbb{Q}(i - (n_Y + d_Y)))$. Remark that thanks to the proof of [Voe03, Thm. 2.4], the natural number n_Y is always positive. The conjecture 1.2 implies now that this group is decomposed into

$$H_{st\mathbb{A}^1}^{2(i-d_Y-n_Y)}(L_+, \mathbb{Q}(i - d_Y - n_Y)) = H_B^{2(i-d_Y-n_Y)}(L_+, \mathbb{Q}(i - d_Y - n_Y)) \oplus H^{i-d_Y-n_Y}(L_+, \mathbf{W}_{\mathbb{Q}}).$$

But $i < d_Y + n_Y$, all these cohomology groups have to vanish, so we can conclude the lemma. \square

The lemma will imply the following

Corollary 5.3. *Under the assumptions as above, one has a canonical isomorphism*

$$H_{\mathbb{A}^1}^{2(d_Y+n_Y)}(X_+ \wedge Th(V_Y/Y), \mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y)) \cong H_{st\mathbb{A}^1}^{2(d_Y+n_Y)}(X_+ \wedge Th(V_Y/Y), \mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y)).$$

Proof. The proof follows immediately from the lemma 5.2 above and the next proposition. \square

Proposition 5.4. [AH11, Prop. 3.3.5] *For any pointed k -space (\mathcal{X}, x) , there is a canonical isomorphism*

$$H_{Nis}^p(\mathcal{X}, \mathbf{K}_p^{MW}) \xrightarrow{\cong} H_{Nis}^{p+1}(\mathcal{X} \wedge \mathbb{P}^1, \mathbf{K}_{p+1}^{MW})$$

Now we are able to finish the proof of the main theorem.

Proposition 5.5. *Let k be an infinite perfect field of $\text{char}(k) \neq 2$. Let $X, Y \in \text{SmProj}/k$ be smooth projective k -schemes. Under the conjectures 1.1 and 1.2 one has a canonical isomorphism*

$$\text{Hom}_{D_{\mathbb{A}^1}(k)_{\mathbb{Q}}}(C_*^{st\mathbb{A}^1}(X), C_*^{st\mathbb{A}^1}(Y)) \cong \widetilde{CH}^{d_Y}(X \times Y, p_Y^{XY*}\omega_{Y/k})_{\mathbb{Q}}.$$

Consequently, $CHW(k)_{\mathbb{Q}}^{op} \rightarrow D_{\mathbb{A}^1, gm}(k)_{\mathbb{Q}}$ is a fully faithful embedding.

Proof. Denote by 0_X the trivial vector bundle of rank 0 on a scheme X . Let us denote by d_X resp. d_Y the dimension of X resp. Y . We have

$$\begin{aligned} \text{Hom}_{D_{\mathbb{A}^1}(k)_{\mathbb{Q}}}(C_*^{st\mathbb{A}^1}(X), C_*^{st\mathbb{A}^1}(Y)) &\stackrel{(1)}{\cong} \text{Hom}_{D_{\mathbb{A}^1}(k)_{\mathbb{Q}}}(C_*^{st\mathbb{A}^1}(X) \otimes C_*^{st\mathbb{A}^1}(Y)^{\vee}, \mathbb{Q}) \\ &\stackrel{(2)}{\cong} \text{Hom}_{D_{\mathbb{A}^1}(k)_{\mathbb{Q}}}(C_*^{st\mathbb{A}^1}(X) \otimes \widetilde{C}_*^{st\mathbb{A}^1}(Th(-T_Y)), \mathbb{Q}) \\ &\stackrel{(3)}{\cong} \text{Hom}_{D_{\mathbb{A}^1}(k)_{\mathbb{Q}}}(\widetilde{C}_*^{st\mathbb{A}^1}(Th(0_X)) \otimes \widetilde{C}_*^{st\mathbb{A}^1}(Th(V_Y)), \mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y)[2(d_Y + n_Y)]) \\ &\stackrel{(4)}{\cong} \text{Hom}_{D_{\mathbb{A}^1}(k)_{\mathbb{Q}}}(\widetilde{C}_*^{st\mathbb{A}^1}(Th(0_X) \wedge Th(V_Y)), \mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y)[2(d_Y + n_Y)]) \\ &\stackrel{(5)}{\cong} \text{Hom}_{D_{\mathbb{A}^1}(k)_{\mathbb{Q}}}(\widetilde{C}_*^{st\mathbb{A}^1}(Th(0_X \times V_Y/X \times Y)), \mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y)[2(d_Y + n_Y)]) \\ &\stackrel{(6)}{\cong} \text{Hom}_{D_{\mathbb{A}^1}^{eff}(k)}(\widetilde{C}_*^{\mathbb{A}^1}(Th(0_X \times V_Y/X \times Y)), \mathbb{Q}_{\mathbb{A}^1}(d_Y + n_Y)[2(n_Y + d_Y)]) \\ &\stackrel{(7)}{\cong} H_{Nis}^{d_Y + n_Y}(Th(0_X \times V_Y/X \times Y), \mathbf{K}_{d_Y + n_Y}^{MW})_{\mathbb{Q}} \stackrel{(8)}{\cong} H_{Nis}^{d_Y}(X \times Y, \mathbf{K}_{d_Y}^{MW}(\det(0_X \times V_Y/X \times Y)))_{\mathbb{Q}} \\ &\stackrel{(9)}{\cong} \widetilde{CH}^{d_Y}(X \times Y, p_Y^{XY*}\omega_{Y/k})_{\mathbb{Q}} \stackrel{(10)}{=} \text{Hom}_{CHW(k)_{\mathbb{Q}}}(Y, X). \end{aligned}$$

We explain now all these identifications. (1) follows from $Hom - \otimes$ adjunction isomorphism 3.2. The second isomorphism (2) is a consequence of duality formalism in \mathbb{P}^1 -stable \mathbb{A}^1 -derived category (Proposition 3.8). The third isomorphism (3) is a consequence of the Theorem 2.4, which will imply an identification $\Sigma_{\mathbb{P}^1}^{\infty}(Th(V_Y)) \xrightarrow{\cong} \Sigma_{\mathbb{P}^1}^{n_Y + d_Y} \Sigma_{\mathbb{P}^1}^{\infty}(Th(-T_Y))$. (one should beware that the notation here is not a suspension spectrum, see [AH11, Exam. 3.1.4] for the discussion). The isomorphisms (4) and (5) follows from definitions. (6) is the corollary 5.3. The identification (7) is the lemma 5.2. The identification (8) is the Thom isomorphism 5.4 in the Theorem 5.1. (9) follows from the simple observation that

$$\det(0_X \times V_Y/X \times Y) \cong \bigwedge_{\mathcal{O}_{X \times Y}} p_Y^{XY*} V_Y \cong p_Y^{XY*} \bigwedge_{\mathcal{O}_Y} V_Y \cong p_Y^{XY*} \omega_{Y/k},$$

where the last isomorphism follows from the property of determinant. (9) is the identification 5.3 between Nisnevich cohomology groups with coefficient as twisted Milnor-Witt K -theory sheaves and twisted Chow-Witt groups and (10) is the definition of morphisms in $CHW(k)$. \square

Remark 5.6. If we assume -1 is a sum of squares in k , the result of F. Morel (cf. [CD10, Cor. 15.2.14]), which says that the stable \mathbb{A}^1 -derived category is equivalent to the category of Beilinson's motives $D_{\mathbb{A}^1}(k)_{\mathbb{Q}} \cong DM_B(k)$, so the result on the embedding in this case is just the usual embedding theorem of Voevodsky.

Acknowledgements : I am very grateful to my advisor H  l  ne Esnault for her encouragement and her patience during the years. I thank Aravind Asok for answering many questions and for explaining to me the Hurewicz formalism, the ε -commutativity of the intersection product in Chow-Witt ring in sense of [Mor12] and the counterexample to the naive contraction of \mathbb{A}^1 -cohomology sheaves. I thank Marc Levine for reading many earlier versions of this note, which contains a lot of mistakes and errors. I heartly thank him for many corrections and suggestions. I thank Jean Fasel for many helpful comments on an earlier version of this note and also for sharing me his proof of the projection formula and his preprint with B. Calm  s on finite Chow-Witt correspondences.

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